MAT 319 - Foundations of Analysis

Final Exam Practice Problems

Stony Brook University
Fall 2023

For practice problems on the *Axiom of Completeness*, *Supremum and Infimum*, *Sequences and Series*, and *Limits of Functions* see the practice problems posted for Midterm I and Midterm II.

Continuous Functions

(1) State the intermediate value theorem.

(2) Prove that the equation $\cos x = \sqrt{x}$ has a solution in the interval $[0, \pi/2]$. If you use some theorem about continuous functions, then check carefully the assumptions.

(3) Prove that the equation $e^{-x} = \sin x$ has a solution in the interval $[0, \infty)$. If you use some theorem about continuous functions, then check carefully the assumptions. Is the solution unique or is there a second solution to the equation?

(4) Suppose that f is a continuous function on $[0, 1]$ with $f(0) = 319$ and such that for each $x \in (0, 1]$ we have $f(x) = 319$ or $f(x) = 320$. Show that $f(x) = 319$ for all $x \in [0, 1]$.

(5) Let $f\colon (a, b) \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to a^+} f(x) = A \quad \text{and} \quad \lim_{x\to b^-} f(x) = B$$

for some $A, B \in \mathbb{R}$ with $A < B$. Show that the interval (A, B) is contained in the range of f. That is, show that for each $k \in (A, B)$ there exists $c \in (a, b)$ such that $f(c) = k$.

(6) Let $f\colon [0, \infty) \to \mathbb{R}$ be a continuous function such that

$$f(0) > 0 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0.$$

Prove that f attains a maximum value. That is, there exists a point $c \in [0, \infty)$ such that $f(x) \le f(c)$ for each $x \in [0, \infty)$.

(7) When do we say that a function is uniformly continuous?

(8) Show that the function

$$g(x) = \frac{1}{5x+3}$$

is uniformly continuous on the interval $[0, \infty)$.

(9) Show that the function

$$g(x) = x^2$$

is uniformly continuous on the interval $[0, 10]$.

(10) Show that the function

$$g(x) = x^2$$

is not uniformly continuous on the interval $[0, \infty)$. In your proof you may use the definition of uniform continuity or some other criterion.

(11) Show that the function $f(x) = \cos(1/x)$ is not uniformly continuous on the interval $(0, 1)$. You can use either the definition of uniform continuity or some other criterion.

(12) Let $f\colon A \to \mathbb{R}$ be a function. Suppose that there exists $L > 0$ with the property that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in A$. Show that f is uniformly continuous.

Derivatives

(1) When do we say that a function is differentiable at a point?

(2) Suppose that $f(x) = x^2$ when x is rational and $f(x) = 0$ when x is irrational. Show that $f'(0)$ exists and compute it.

(3) Compute each of the following limits if it exists and otherwise explain why it does not exist. If you apply some theorem, then explain why it can be applied.

$$\lim_{x \to 0^+} \frac{1}{x(\ln x)^2}$$

$$\lim_{x \to 0^+} (\ln(x+1))^x$$

$$\lim_{x \to \infty} \frac{2\cos x + x}{\sin x + 2x}$$

(See the textbook for further problems on L'Hospital's rule.)

(4) State the mean value theorem.

(5) Suppose that $f'(x) > 0$ for each $x \in (a, b)$. Use the mean value theorem to show that f is strictly increasing on (a, b).

(6) Let $f\colon [a, b] \to \mathbb{R}$ be a differentiable function. Suppose that there exists $L > 0$ such that $|f'(x)| \leq L$ for each $x \in [a, b]$. Show that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$. Then show that f is uniformly continuous.

(7) Suppose that $f\colon (a,b) \to \mathbb{R}$ is a differentiable function that is uniformly continuous. Is f' a bounded function? If yes, then provide a proof. If no, then give a counterexample.

(8) Consider the function

$$f(x) = |x^2 - 1|,$$

where $x \in \mathbb{R}$. Is this function continuous at $x = 1$? Moreover, is it differentiable at $x = 1$? Prove your claims either using some theorem or using the definitions.

(9) Explain why the function $f(x) = |x^2 - 1|$, $x \in [-2, 3]$, has an absolute maximum and an absolute minimum and then find them.

(10) Consider the function $f(x) = \ln x$, $x > 0$. For each $n \in \mathbb{N}$, find a formula for the n-th derivative $f^{(n)}(x)$. Then use Taylor's theorem with center at the point 1 to approximate the number $\ln 2$ up to an error of 10^{-1}. Explain carefully. You can leave your answer as a sum of finitely many rational numbers.

Integrals

(1) When do we say that a function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable?

(2) Show that the function

$$g(x) = \begin{cases} 1, & 3 \leq x < 5 \\ 10, & x = 5 \end{cases}$$

is Riemann integrable in the interval $[3, 5]$ by using only the definition.

(3) Suppose that $f\colon [0,1] \to \mathbb{R}$ is a Riemann integrable function with $f(x) \geq 0$ for all $x \in [0,1]$, and f is not identically the zero function. Is it true that

$$\int_0^1 f(x)\,dx > 0?$$

(4) Suppose that f is continuous on $[a,b]$, $f \geq 0$, and $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$.

(5) If $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable and $|f(x)| \leq M$ for all $x \in [a,b]$, show that

$$\left| \int_a^b f(x)\,dx \right| \leq M(b-a).$$

(6) Let $f(x) = 0$ when x is rational and $f(x) = 1$ when x is irrational. Show that f is not Riemann integrable on $[0,1]$.

(7) Using only the definition or Cauchy's criterion for integrability or some divergence criterion, show that the function

$$h(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is not Riemann integrable on $[0,1]$.